Relativistic theories of interacting fields and fluids

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Abstract

We investigate divergence-type theories describing the disipative interaction betwen a field and a fluid. We look for theories which, under equilibrium conditions, reduce to the theory of a Klein-Gordon scalar field and a perfect fluid. We show that the requirements of causality and positivity of entropy production put non-trivial constraints to the structure of the interaction terms. These theories provide a basis for the phenomonological study of the reheating period.

I. INTRODUCTION

In this paper we investigate divergence-type theories (DTT) describing the dissipative interaction between a field and a fluid [1,2]. We look for theories which, under equilibrium conditions, reduce to the theory of a Klein-Gordon (KG)scalar field and a perfect fluid. We show that the requirements of causality and positive entropy production put non-trivial constraints on the structure of the interaction terms.

The motivation for this work comes from the development of inflationary cosmological models [3–5]. In these models of the Early Universe, most of the energy density is concentrated in a single (fundamental or effective) scalar field or "inflaton". There are two well defined moments in the evolution, the roll - down period and the reheating period. In the former the inflaton provides an effective cosmological constant and supports the superluminal expansion of the Universe [6]; in the later the inflaton decays into ordinary matter, thus creating entropy and heating up the Universe.

While inflationary models assume a simple spatial dependence for both the inflaton and the metric of the Universe, scalar and gravitational fluctuations around this simple background play an important role

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[7]. During roll - down, scalar and tensor fluctuations are redshifted and frozen by the cosmological expansion, and become the seeds for future contrasts in the energy density of the Universe. During reheating, fluctuations within the horizon interact in a complex way with the background fields, and are key to such questions as what is the duration of the reheating period, at which temperature is equilibrium attained, how is energy distributed at the end of reheating, and the spectrum of primordial density fluctuations.

Overall, quantum field theory in curved spaces is an adequate framework to study the roll - down period (although of course many questions remain, such as what is the shape of the inflaton potential, and whether the fine tuning usually assumed in standard inflationary models is truly unavoidable) [8–10]. However, reheating is much more complex, as here neither background fields nor fluctuations may be considered a perturbation on the other, and moreover the way the geometry changes along the process has a drastic influence on results. For this reason, analysis based on quantum field theory on curved spaces have been satisfactory only for the early stages of reheating, the so-called preheating where the dominant process is particle creation from the background field through parametric amplification [11–18].

To the best of our knowledge, the most successful strategy to deal with reheating in the strongly nonlinear regime has been to describe the inflaton field(s) as purely classical [19]. By modelling the inflaton as a classical field obeying a non linear wave equation it is possible to go beyond perturbation theory; moreover, this approach may be justified on the basis that occupation numbers, in the infrared modes of interest, are typically very high. However, a purely classical theory ignores virtual processes that may modify the behavior of the quantum theory, as has been shown by recent detailed calculations of transport coefficients from quantum field theory [20–25].

The next step in improving classical models of reheating is to incorporate those quantum aspects as phenomenological terms within the classical theory [26]. For example, the inflaton and its fluctuations may be depicted as a classical dissipative fluid, with constitutive relationships derived in some way from quantum field theory.

The simplest and first approach to reheating, for example, is to transform the Klein - Gordon equation into a telegraphist's equation by adding a $\Gamma\dot{\phi}$ dissipative term, where Γ is estimated from the quantum mechanical decay rate of the inflaton ([27–30]). As pointed out by Brandenberger and others [31], this approach misses some important components of the reheating process, such as the possibility of preheating, namely, enhanced decay through parametric amplification of quantum fields coupled to the inflaton. A subtler criticism is that this approach, or rather, simple covariant generalizations of this approach, are

equivalent to writing a first order dissipative theory for the inflaton field (in the classification of Hiscock and Lindblom [32]), similar to the Eckart theory for ordinary fluids. However, it is known that first order theories have stability and causality problems, and therefore we should expect the same problems will occur in these simplified reheating scenarios ([33]).

The task at hand is then to write down classical dissipative models of reheating, in order to account for the basic aspects of the quantum phenomenology, but to do so in the framework of a consistent relativistic hydrodynamics, thus building into the model the general principles of relativistic invariance, stability, causality and the Second Law of Thermodynamics from scratch.

In an important precedent to this work, for example, Maartens, Pavón, Zimdahl and others [34–40] have developed a dissipative model of reheating based on the ”truncated” Israel - Stewart framework. However, these authors treated the inflaton as a fluid, thus losing the aspects of the problem associated with the coherence of the inflaton field. Besides, they focused on the dissipative effects arising from the fact that a mixture of otherwise ideal fluids will generally develop a nonvanishing bulk viscosity. Therefore, in their model dissipation happens only if the Universe expands. On the other hand, quantum fields show dissipative behavior even in Minkowski space time. In our work, we shall not only treat the inflaton as a coherent field, but also focus on dissipation arising from having the inflaton not in equilibrium with the ensemble of all other (quantum) fields. In fact, in this preliminary investigation we shall work on flat space time backgrounds, although a fully covariant generalization will be immediate. It is likely that a truly realistic account of reheating will require a combination of both this and the Maartens - Pavón - Zimdahl approaches.

Another important difference between this and earlier work is that, instead of the Israel - Stewart [41–43] framework (or similar frameworks, such as ”Extended Thermodynamics” [44]) , we shall work within the class of ”Divergence Type Theories” (DTT) earlier introduced by Geroch [1]. DTTs are appealing because they represent a mathematically consistent, closed system, rather than a truncated expansion in deviations from ideal behavior. This makes the discussion of such properties as causality and stability most transparent [1,45], for which reason DTTs are a natural language to describe the complex nonlinear interactions between the inflaton field and the fluid of quantum fluctuations during reheating.

To summarize, in this paper we explore DTTs describing the interaction between a classical, nonlinear scalar field and a perfect fluid. Our main goal is to study the constraints on possible interactions which follow from the requirements of covariance, causality, stability and the Second Law. For simplicity, we

shall work on a flat space - time, and when no loss of generality is involved, we shall assume homogeneous configurations. The linearized dynamics of the scalar field around equilibrium is given by the telegraphist's equation, but the characteristic $\Gamma\dot{\phi}$ dissipative term appears only as a linear and local approximation to a more complex term (as it happens in dissipative equations derived from quantum field theory [46,47]).

We shall begin our analysis by constructing a DTT whose solutions, under suitable restrictions on initial conditions, reduce to solutions of a nonlinear Klein - Gordon (KG) equation. Let us note that while it is easy to show that the KG equation is hyperbolic ([48]), to the best of our knowledge this equation was never formulated as a DTT. The dynamics of our DTT theory is given in terms of the conservation equations for the energy momentum tensor and one vector current j_a, and therefore includes one vectorial and one scalar Lagrange multipliers $\beta^a_{(c)}$ and ξ, respectively. Together with the scalar field ϕ, these are the degrees of freedom of the theory.

This is in contradistinction with the usual view of ϕ and its gradient $\phi_{,a}$ as the only degrees of freedom. The extra variable is associated to dissipation and will become a true dynamical variable only when interaction with other fluids is introduced. The fact that dissipative theories require more degrees of freedom than ideal theories is of course rather generic.

Our next step is to introduce the ensemble of quantum fluctuations of all other matter fields. For simplicity, we shall describe this as a single ideal fluid. We shall assume there are no conserved currents other than energy - momentum, so this fluid will introduce only one new vectorial degree of freedom $\beta^a_{(q)}$, representing the inverse temperature vector.

The interacting theory is different from the decoupled theory in two ways. First, the dependence of the energy momentum tensors for each component (field and fluid) on the dynamical variables is not the same (in a derivation from first principles, this would result from taking the variation of interaction and radiative correction terms in the quantum effective action with respect to the metric [49]). For simplicity, we shall assume that the energy - momentum tensors, but not the current j^a, are modified in this way. Second, only the total energy momentum is conserved, so the divergence of the energy - momentum tensor of the field alone, say, is not zero. We may also modify the Klein - Gordon equation (near equilibrium, this modification will result in the addition of a $\Gamma\dot{\phi}$ dissipative term). As it will turn out, the possible modifications in the energy momentum tensors are constrained by causality, while the nonconservation terms in the equations of motion are constrained by the Second Law.

After analyzing the constraints derived from linear stability and causality, we shall conclude this paper

by displaying the simplest possible family of theories satisfying all the physical requirements above. We shall use these theories for a qualitative discussion of the nonlinear aspects of the approach to equilibrium.

The rest of the paper is organized as follows. In next section, we show a well defined DTT which reduces to the KG equation if suitable boundary conditions are enforced. In Section III we study the mixture of field and fluid in the DTT framework. We analyze the equilibrium states and their linear stability. In Section IV we write down a simple, acceptable, nonlinear DTT theory, and use it to analyze the thermalization process. We summarize the main conclusions in some brief final remarks.

We gather in the appendix some elementary facts about DTTs and their connection to causality.

II. DIVERGENCE TYPE THEORY OF THE KLEIN GORDON FIELD

To investigate properly a relativistic fluid, the context of DTTs proposed by Geroch is particularly interesting since causality is easily investigated and/or achieved. We use signature $(-,+,+,+)$, latin letters $a, b, c, ...$ indicate spacetime coordinates and Greek letters $\mu, \nu, ...$ indicates spatial coordinates, other conventions follows Misner, Thorne and Wheeler ([50]). We will also follow the usual (physicist's) convention to designate tensorial character by its elements, that is T^{ab} will designate the tensor $T = T^{ab} e_a \otimes e_b$, context indicating easily whether one is making a statement about the whole tensor or its coordinates.

Our first goal is to derive a model which contains the classical Klein-Gordon field as particular case. Since we want to be able to apply this to the early universe, we allow the classical field to be subject to a non-trivial potential (which may even be non-renormalizable [8–10]). The Klein - Gordon theory may be described as a conservation law for the "current" $\phi_{,a}$, namely, $\nabla^a \phi_{,a} = V'(\phi)$. The energy - momentum tensor is also defined in terms of $\phi_{,a}$ as $T_{ab} = \phi_{,a}\phi_{,b} - g_{ab}\left(\phi_{,c}\phi^{,c}/2 + V(\phi)\right)$. We therefore postulate that our theory is defined by two currents, energy - momentum T_{ab} and a vector current j^a, with dynamical equations

$$T^{ab}{}_{;b} = 0 \tag{1}$$

$$j^a{}_{;a} = R\left[x\right] \tag{2}$$

and the constitutive relation

$$T^{ab} = j^a j^b - g^{ab}\left(\frac{1}{2} j_c j^c + T\left[R\right]\right) \tag{3}$$

The scalar field ϕ is introduced by writing the functional relationship between R and T parametrically as $R = V'(\phi)$, $T = V(\phi)$. This implies no loss of generality.

Next consider the conservation law

$$T^{ab}{}_{;b} = g^{ab}\left(j_{b\,,c} - j_{c\,,b}\right) j^c + V'\left(j^a - \phi^{,a}\right) = 0 \tag{4}$$

Then if

$$j_{b\,,c} - j_{c\,,b} = 0 \tag{5}$$

we have $j^a = \phi^{,a}$ and we fall back on the usual (classical) Klein-Gordon theory. Note that equation (5) is a constraint rather than a dynamical equation. Defining $M_{ab} \equiv j_{a;b} - j_{b;a}$ we have the following identity

$$j^c M_{ab;c} = V' M_{ba} + \frac{V''}{V'} j^c \left(j_b M_{ac} - j_a M_{bc}\right) + j^c{}_{;a} M_{bc} - j^c{}_{;b} M_{ac} \tag{6}$$

Therefore if (5) is true initially, it will stay true for all time. In other words, our set of equations (1), (2) and (3) represent a theory larger than Klein-Gordon, reducing to it if the constraint (5) is enforced initially.

Our next step is to cast this theory within the DTT framework. Since there are two currents, we introduce two Lagrange multipliers $\xi, \beta^a_{(c)}$ (we also define $\beta_{(c)} = \sqrt{-\beta^a_{(c)}\beta_{(c)a}}$) as dynamical degrees of freedom. ξ is analogous to a chemical potential conjugated to the current j^a, while $\beta^a_{(c)}$ plays the role of "inverse temperature" and is conjugated to T^{ab}. Observe the perfect-fluid form of the energy-momentum tensor T^{ab}.

Following the general DTT construction (see Appendix), we introduce the generating function $\chi^a = \beta^a_{(c)} p$, where p is the pressure and

$$j^a = \beta^a_{(c)} \frac{\partial p}{\partial \xi} \tag{7}$$

$$T^{ab} = p g^{ab} - \beta^a_{(c)}\beta^b_{(c)} \frac{1}{\beta_{(c)}} \frac{\partial p}{\partial \beta_{(c)}} = p g^{ab} - j^a j^b \frac{1}{\beta_{(c)}} \frac{\left(\frac{\partial p}{\partial \beta_{(c)}}\right)}{\left(\frac{\partial p}{\partial \xi}\right)^2} \tag{8}$$

leading to the following equation upon comparison with (3)

$$-\frac{1}{\beta_{(c)}} \frac{\partial p}{\partial \beta_{(c)}} = \left(\frac{\partial p}{\partial \xi}\right)^2$$

$$p = -\frac{1}{2} j_c j^c - T \tag{9}$$

The first one is a differential equation in the variables $\xi, \beta_{(c)}$ with solution

$$p = \frac{\xi^2}{2\beta_{(c)}^2} + \Lambda \tag{10}$$

where Λ is independent of ξ and $\beta_{(c)}$ but can depend on spacetime coordinates. In fact, using the second equation we see that

$$\Lambda = -V(\phi) \tag{11}$$

Thus

$$j^a = \frac{\xi}{\beta_{(c)}^2}\beta_{(c)}^a \tag{12}$$

Recall that the conservation law for T^{ab}, eq. (4) implies $j^2 - j^a\phi_{,a} = 0$ even when the constraint eq. (5) is not enforced. Since $j^2 = -(\xi/\beta_{(c)})^2$, we get

$$\beta_{(c)}^a \phi_{,a} = -\xi \tag{13}$$

Finally, we easily compute χ_c using

$$p = -\frac{1}{\beta_{(c)}}\frac{\partial \chi_c}{\partial \beta_{(c)}} \tag{14}$$

Leading immediately to

$$\chi_c = -\frac{1}{2}\xi^2 \ln \beta_{(c)} + \frac{1}{2}T\beta_{(c)}^2 \tag{15}$$

We can regard eq. (13) as a generalization of the canonical momentum $\pi = \dot{\phi}$; in the "rest" frame where $\beta_{(c)}^a = \left(\beta_{(c)}, \vec{0}\right)$, we get $\pi = -\xi/\beta_{(c)}$. The problem is that only this ratio has a direct meaning in terms of the Klein - Gordon theory *alone*, that is, the KG equations are invariant under a rescaling of ξ and $\beta_{(c)}^a$ by a common factor. In order to break this indeterminacy, we must look at the larger framework where the Klein - Gordon field interacts with other fluids. Then we complete the definition of ξ and $\beta_{(c)}^a$ by demanding that, in equilibrium, $\beta_{(c)}^a$ must be identical to the (only) inverse temperature vector of the full theory.

Knowing $\beta_{(c)}^a$, we now regard eq. (13) as the *definition* of the scalar ξ. This means that, while for the pure KG theory eq. (13) simply follows from energy - momentum conservation, we shall demand it also holds unchanged in the interacting theory. This procedure is of course suggested by Landau and Lifschitz' treatment of the damped harmonic oscillator in ref. [51]

Since χ_c^a is a homogeneous function of degree 1 in ξ and $\beta_{(c)}^a$, the entropy current and entropy creation rate vanish in the pure KG theory, as expected for a coherent field.

III. DIVERGENCE TYPE THEORY OF INTERACTING FIELDS AND FLUIDS

Our next goal is to describe the interaction between the KG field and other forms of matter in the context of DTTs. We thus introduce a (perfect) fluid, described by an inverse temperature vector $\beta^a_{(q)}$ and energy - momentum tensor T^{ab}_q derivable from a generating functional $\chi^a_q = \partial\chi_q/\partial\beta_{(q)\,a}$ (see the Appendix). The interacting theory shall be described by the equations

$$\begin{aligned} j^a_{\ ;a} &= R + \Delta \\ T^{ab}_{c\ ;b} &= I^a \\ T^{ab}_{q\ ;b} &= -I^a \\ \beta^a_{(c)}\phi_{,a} &= -\xi \end{aligned} \tag{16}$$

Where j^a and T^{ab}_c are the current and energy - momentum tensor for the inflaton field, and we have added the definition eq. (13), which, unlike the situation in the noninteracting theory, is now independent of the other equations . Let us seek a generating functional of the form

$$\chi^a = \chi^a_c + \chi^a_q + \Xi^a \tag{17}$$

The total system is generated not only by the sum of each thermodynamic potential but there is a third potential to include the interaction between field and fluid. Each energy - momentum tensor will be given by:

$$\begin{aligned} T^{ab}_c &= \frac{\partial\chi^a_c}{\partial\beta_{(c)\,b}} + \frac{\partial\Xi^a}{\partial\beta_{(c)\,b}} \\ T^{ab}_q &= \frac{\partial\chi^a_q}{\partial\beta_{(q)\,b}} + \frac{\partial\Xi^a}{\partial\beta_{(q)\,b}} \end{aligned} \tag{18}$$

Let's define the following variables

$$\begin{aligned} \beta^a &= \frac{1}{2}\left(\beta^a_{(c)} + \beta^a_{(q)}\right) \\ B^a &= \beta^a_{(c)} - \beta^a_{(q)} \end{aligned} \tag{19}$$

Then

$$\begin{aligned} \frac{\partial\Xi^a}{\partial\beta_{(c)\,b}} &= \frac{1}{2}\frac{\partial\Xi^a}{\partial\beta_b} + \frac{\partial\Xi^a}{\partial B_b} \\ \frac{\partial\Xi^a}{\partial\beta_{(q)\,b}} &= \frac{1}{2}\frac{\partial\Xi^a}{\partial\beta_b} - \frac{\partial\Xi^a}{\partial B_b} \end{aligned} \tag{20}$$

Thus

$$T_c^{ab} + T_q^{ab} = \frac{\partial \chi_c^a}{\partial \beta_{(c)\,b}} + \frac{\partial \chi_q^a}{\partial \beta_{(q)\,b}} + \frac{\partial \Xi^a}{\partial \beta_b} \tag{21}$$

Note that the only real physical system is the one described by the total energy-momentum tensor given above. Any separation in two fluids will be tinged with arbitrariness, a fact that will be clear soon. We will ask that the total energy-momentum tensor be symmetric; therefore

$$\Xi^a = \frac{\partial \Xi}{\partial \beta_a} \tag{22}$$

Ξ will depend in general on scalars as demanded by Lorentz invariance; namely $\Xi = \Xi(\xi, u, v, w, \phi)$ where

$$\begin{aligned} u &= -\beta_a \beta^a \\ v &= -B_a B^a \\ w &= -\beta_a B^a \end{aligned} \tag{23}$$

Then

$$\begin{aligned} \frac{\partial}{\partial \beta_a} &= -2\beta^a \frac{\partial}{\partial u} - B^a \frac{\partial}{\partial w} \\ \frac{\partial}{\partial B_a} &= -2B^a \frac{\partial}{\partial v} - \beta^a \frac{\partial}{\partial w} \end{aligned} \tag{24}$$

Note that even if the source term I^b is taken to be null, the energy momentum tensors for field and fluid do not fall back automatically to their old form. Since the only "true" energy-momentum tensor is the total one, it is sometimes helpful and instructive to rewrite the equation of motions of the T_i^{ab} $(i = c, q)$. Let us define the following

$$\begin{aligned} T_+^{ab} &= T_c^{ab} + T_q^{ab} \\ T_-^{ab} &= T_c^{ab} - T_q^{ab} \end{aligned} \tag{25}$$

Note that

$$T_+^{ab} = \frac{\partial \chi^a}{\partial \beta_b} \tag{26}$$

since $\partial \chi_q^a / \partial \beta_{(c)\,b} = 0 = \partial \chi_c^a / \partial \beta_{(q)\,b}$. Also

$$T_-^{ab} = 2 \frac{\partial \chi^a}{\partial B_b} \tag{27}$$

The following identities follow straightforwardly

$$\Xi^a = -2\beta^a \frac{\partial \Xi}{\partial u} - B^a \frac{\partial \Xi}{\partial w} \tag{28}$$

$$\frac{\partial \Xi^a}{\partial \beta_b} = -2g^{ab}\frac{\partial \Xi}{\partial u} + 4\beta^a\beta^b\frac{\partial^2 \Xi}{\partial u^2} + B^a B^b \frac{\partial^2 \Xi}{\partial w^2} + 2\left(\beta^a B^b + B^a \beta^b\right)\frac{\partial^2 \Xi}{\partial u \partial w} \tag{29}$$

$$\frac{\partial \Xi^a}{\partial B_b} = -g^{ab}\frac{\partial \Xi}{\partial w} + 2\beta^a\beta^b\frac{\partial^2 \Xi}{\partial u \partial w} + 2B^a B^b\frac{\partial^2 \Xi}{\partial v \partial w} + 4\beta^a B^b\frac{\partial^2 \Xi}{\partial u \partial v} + B^a\beta^b\frac{\partial^2 \Xi}{\partial w^2} \tag{30}$$

Let 's compute the entropy creation rate. Using

$$\begin{aligned} \frac{\partial}{\partial \beta_b} &= \frac{\partial}{\partial \beta_{(c)\,b}} + \frac{\partial}{\partial \beta_{(q)\,b}} \\ \frac{\partial}{\partial B_b} &= \frac{1}{2}\frac{\partial}{\partial \beta_{(c)\,b}} - \frac{1}{2}\frac{\partial}{\partial \beta_{(q)\,b}} \end{aligned} \tag{31}$$

we find

$$\nabla_a S^a = \frac{\partial \Xi^a}{\partial \phi}\nabla_a \phi - B_b I^b - \xi \Delta \tag{32}$$

where we used the fact that $\beta^a_{(c)}\phi_{,a} + \xi = 0$. For simplicity, let us ask that Ξ be independent of ϕ, so that the entropy production reduces to

$$\nabla_a S^a = -B_b I^b - \xi \Delta \tag{33}$$

The second law of thermodynamics imposes that $\nabla_a S^a > 0$. This means that I^b and Δ must vanish when B^b and ξ go to zero. In this limit, Δ, which is a scalar, must take the form $\Delta = A\xi + Bw$ (v being of higher order); Lorentz invariance demands $I^b = -\left(C\xi + Dw\right)\beta^b - EB^b$. Therefore $\nabla_a S^a = -A\xi^2 - \left(B + C\right)w\xi - Dw^2 - Ev$, and we must have $A, D \leq 0$, $E \leq 0$ and $AD \geq \left(B + C\right)^2/4$. Equivalently, we may parametrize $A = M^{\xi\xi}$, $B\beta^a = 2(1-\kappa)M^{\xi a}$, $C\beta^b = 2\kappa M^{\xi b}$ and $D\beta^a\beta^b + Eg^{ab} = M^{ab}$, whereby

$$\begin{aligned} \Delta &\equiv M^{\xi\xi}\xi + 2B_a(1-\kappa)M^{\xi a} \\ I^b &\equiv 2\kappa\xi M^{\xi b} + B_a M^{ab} \end{aligned} \tag{34}$$

A. First order analysis away from equilibrium in a simplified, homogeneous model

Let 's turn now to the equation of motion to analyze small deviations from equilibrium. The requirement that the equilibrium state must be isotropic in some "rest" frame implies that, in equilibrium, the two

vector $\beta^a_{(c)}$ and $\beta^a_{(q)}$ must be parallel. Therefore we can write $\beta^a_{(c)} = \beta_c u^a$ and $\beta^a_{(q)} = \beta_q u^a$, with a common unit vector u^a. We will also restrict ourselves to the homogeneous case. Observe that $T^{\mu b}_{c\ ;b} = T^{\mu b}_{q\ ;b} = 0$ identically for $\mu \neq 0$, so there are only four nontrivial equations, including eq. (13).

In equilibrium, we must have $\xi = \Delta = I^a = R = 0$. Defining $\phi = 0$ at the equilibrium point, we conclude $(\xi_{eq}, \beta_{eq}, B_{eq}, \phi_{eq}) = (0, \beta_0, 0, 0)$. We can now analyze the first order deviations away from equilibrium; that is

$$
\begin{aligned}
\xi &= \delta\xi \\
\phi &= \delta\phi \\
\beta &= \beta_{eq} + \delta\beta \\
B &= \delta B
\end{aligned}
\tag{35}
$$

We can expand

$$
\Xi = \sum_{n=0}^{\infty} F_n(\beta, \xi)\, B^{n+2} \tag{36}
$$

Ξ is constrained to this form by demanding that at equilibrium, where $\beta_c = \beta_q$, we fall back to our initial T^{ab} and j^a. Δ and I^b are given by eq. (34), where all the matrix element are evaluated at the equilibrium value $\xi_{eq} = B^a_{eq} = 0$, $\beta^a_{eq} = \beta^{eq}_0$ and $\phi_0 = 0$. Moreover we will define $m^2 = V''(0)$, so to first order

$$
R = m^2 \delta\phi \tag{37}
$$

In our case we have

$$
\begin{aligned}
j^a_{\ ;a} &= \frac{1}{\beta_0} (\delta\xi)_{,t} = M_0^{\xi\xi} \delta\xi + 2(1-\kappa) M_0^{\xi 0} \delta B + m^2 \delta\phi \\
T^{0b}_{c\ ;b} &= -\frac{\partial^3 \Xi}{\partial\beta \partial B^2} (\delta B)_{,t} = 2\kappa M_0^{\xi 0} \delta\xi + M_0^{00} \delta B \\
T^{0b}_{q\ ;b} &= \rho'_q (\delta\beta)_{,t} + \left(\frac{\partial^3 \Xi}{\partial\beta \partial B^2} - \frac{1}{2}\rho'_q \right) (\delta B)_{,t} = -2\kappa M_0^{\xi 0} \delta\xi - M_0^{00} \delta B \\
\beta_0 (\delta\phi)_{,t} &= -\delta\xi
\end{aligned}
\tag{38}
$$

We can revert to our old coordinates $\delta\beta_c$ and $\delta\beta_q$

$$
\frac{1}{\beta_0} (\delta\xi)_{,t} = M_0^{\xi\xi} \delta\xi + 2(1-\kappa) M_0^{\xi 0} \delta\beta_c + 2(1-\kappa) M_0^{\xi 0} \delta\beta_q + m^2 \delta\phi
$$

$$
\begin{aligned}
-\frac{\partial^3\Xi}{\partial\beta\partial B^2}\left(\delta\beta_c\right)_{,t}+\frac{\partial^3\Xi}{\partial\beta\partial B^2}\left(\delta\beta_q\right)_{,t} &= 2\kappa M_0^{\xi 0}\delta\xi+M_0^{00}\delta\beta_c-M_0^{00}\delta\beta_q \\
\left(\frac{\partial^3\Xi}{\partial\beta\partial B^2}\right)\left(\delta\beta c\right)_{,t}+\left(\rho_q'-\frac{\partial^3\Xi}{\partial\beta\partial B^2}\right)\left(\delta\beta_q\right)_{,t} &= -2\kappa M_0^{\xi 0}\delta\xi-M_0^{00}\delta\beta_c+M_0^{00}\delta\beta_q \\
\beta_0\left(\delta\phi\right)_{,t} &= -\delta\xi
\end{aligned}
\tag{39}
$$

The advantage to do this is that we can add the middle two to obtain

$$
\rho_q'\left(\delta\beta_q\right)_{,t}=0 \tag{40}
$$

That is

$$
\delta\beta_q=constant \tag{41}
$$

We are thus left with

$$
\begin{aligned}
\frac{1}{\beta_0}\left(\delta\xi\right)_{,t} &= M_0^{\xi\xi}\delta\xi+2(1-\kappa)M_0^{\xi 0}\delta\beta_c+m^2\delta\phi \\
-\frac{\partial^3\Xi}{\partial\beta\partial B^2}\left(\delta\beta_c\right)_{,t} &= 2\kappa M_0^{\xi 0}\delta\xi+M_0^{00}\delta\beta_c \\
\beta_0\left(\delta\phi\right)_{,t} &= -\delta\xi
\end{aligned}
\tag{42}
$$

or, in matrix form

$$
\begin{pmatrix} \left(\beta_0\right)^{-1} & 0 & 0 \\ 0 & -\frac{\partial^3\Xi}{\partial\beta\partial B^2} & 0 \\ 0 & 0 & \beta_0 \end{pmatrix}
\begin{pmatrix} \delta\xi \\ \delta\beta_c \\ \delta\phi \end{pmatrix}_{,t}
=
\begin{pmatrix} M_0^{\xi\xi} & 2(1-\kappa)M_0^{\xi 0} & m^2 \\ 2\kappa M_0^{\xi 0} & M_0^{00} & 0 \\ -1 & 0 & 0 \end{pmatrix}
\begin{pmatrix} \delta\xi \\ \delta\beta_c \\ \delta\phi \end{pmatrix}
\tag{43}
$$

We will consider solutions of the type $\exp\left(-\gamma t\right)$. There is a solution if

$$
\begin{vmatrix} \left(-\gamma\beta_0^{-1}-M_0^{\xi\xi}\right) & -2(1-\kappa)M_0^{\xi 0} & -m^2 \\ -2\kappa M_0^{\xi 0} & \left(\gamma\frac{\partial^3\Xi}{\partial\beta\partial B^2}-M_0^{00}\right) & 0 \\ 1 & 0 & -\gamma\beta_0 \end{vmatrix}=0
\tag{44}
$$

To make notation less clumsy, let us write

$$
\begin{aligned}
M_0^{\xi\xi} &= \varepsilon_1=A \\
\frac{\partial^3\Xi}{\partial\beta\partial B^2} &= X \\
M_0^{\xi 0} &= \varepsilon_2=\left(B+C\right)\frac{\beta^0}{2} \\
M_0^{00} &= \varepsilon_3=D\beta_0^2+E
\end{aligned}
\tag{45}
$$

The notation is chosen to emphasize explicitly which quantities are small and which aren't . Namely, the ε_i are (very) small but the X is not. Therefore

$$\begin{vmatrix} \left(-\gamma\beta_0^{-1} - \varepsilon_1\right) & -2\left(1-\kappa\right)\varepsilon_2 & -m^2 \\ -2\kappa\varepsilon_2 & \left(\gamma X - \varepsilon_3\right) & 0 \\ 1 & 0 & -\gamma\beta_0 \end{vmatrix} = 0 \tag{46}$$

Expanding with the second column we have

$$4\kappa\left(1-\kappa\right)\gamma\beta_0\varepsilon_2^2 + X\beta_0\gamma^2\varepsilon_1 - \beta_0\gamma\varepsilon_1\varepsilon_3 + X\gamma\left(\gamma^2+m^2\right) - \varepsilon_3\left(\gamma^2+m^2\right) = 0 \tag{47}$$

The solutions to order zero are $\gamma = \pm im$ and $\gamma = 0$. To order one, the equation reduces to

$$X\beta_0\gamma^2\varepsilon_1 + X\gamma\left(\gamma^2+m^2\right) - \varepsilon_3\left(\gamma^2+m^2\right) = 0 \tag{48}$$

Writing first $\gamma = \pm im + \delta$ we obtain

$$\delta = -\frac{\beta_0\varepsilon_1}{2} \tag{49}$$

and in the case $\gamma = \delta$ we have

$$\delta = \frac{\varepsilon_3}{X} \tag{50}$$

That is

$$\begin{aligned} \gamma &= \pm im - \frac{\beta_0\varepsilon_1}{2} \\ \gamma &= \frac{\varepsilon_3}{X} \end{aligned} \tag{51}$$

Notice that $\varepsilon_1 < 0$ and $\varepsilon_3 < 0$. Since $\beta_0 > 0$ then the first two solutions corresponding to oscillatory modes get a damping part. The other solution will also be damped if

$$X \equiv \frac{\partial^3\Xi}{\partial\beta\partial B^2} < 0 \tag{52}$$

In this case the system will be causal if the matrix $M^{0AB}v$ is negative-definite (where $v = v^0 > 0$ and $v_0 = -v^0$, since it is the temporal component of a temporal 4-vector oriented toward the future) and where

$$M^{0AB}v = \begin{pmatrix} -v\beta_0^{-1} & 0 & 0 \\ 0 & v\frac{\partial^3\Xi}{\partial\beta\partial B^2} & 0 \\ 0 & 0 & -v\beta_0 \end{pmatrix}$$

Obviously, this matrix is negative-definite if

$$\beta_0 > 0$$
$$\frac{\partial^3 \Xi}{\partial\beta\partial B^2} < 0$$

We thus verify that the causality condition implies the stability of the equilibrium solution. This is in agreement with the findings of Hiscock and Lindblom in the context of the Israel - Stewart formalism [32].

Going one order further we find

$$\gamma = \pm i \left\{ m - \frac{\beta_0}{2m} \left[\frac{1}{4}\beta_0 \varepsilon_1^2 - \frac{4\kappa(1-\kappa)}{X}\varepsilon_2^2 \right] \right\} - \frac{\beta_0 \varepsilon_1}{2} \tag{53}$$

$$\gamma = \frac{\varepsilon_3}{X} \tag{54}$$

That is the correction to the third solution is at least third order and the second order terms modify the frequency of the oscillation in the case of the other two solutions.

If we look back to the linearized equation of motion for $\delta\phi$, we obtain

$$-\left(\delta\phi\right)_{,tt} = -M_0^{\xi\xi}\beta_0 \left(\delta\phi\right)_{,t} + 2(1-\kappa)M_0^{\xi 0}\delta\beta_c + m^2\delta\phi \tag{55}$$

In the simplest case $M_0^{\xi 0} = 0$, this is the telegraphist equation with a damping term $\Gamma\dot{\phi}$, where $\Gamma = \left|M_0^{\xi\xi}\right|\beta_0$. It is important to realize, however, that this identification holds only to linear order away from equilibrium. In general, Γ will not be a constant, but a function of the dynamical variables β_q and β_c, and therefore it will depend on the history of the system.

We may observe that when $M_0^{\xi 0} = 0$, we get $\beta_{c,t} = 0$ to first order.

IV. A SIMPLE NONLINEAR MODEL OF FIELD - FLUID INTERACTION

In this section we shall investigate the simplest DTTs of field - fluid interaction satisfying the requirements of causality, stability and the Second Law discussed in the previous Section. To make things simplest, we will restrict the form of Ξ in the following manner

$$\Xi = F(u)v + G(u)w^2 \tag{56}$$

and we will investigate what restrictions one can expect from causality on F and G. By making Ξ independent of ξ, we make sure that the current j^a preserves its form in the interacting theory. We also

assume Ξ to be $\phi-$ independent, and require it to be quadratic on the difference variable B^a, in such a way that corrections to the energy - momentum tensors will vanish in the equilibrium state. Observe that

$$\Xi^a = -2\beta^a\left(F'v + G'w^2\right) - 2B^a G w$$

$$\frac{\partial \Xi^a}{\partial \beta_b} = 4\beta^a\beta^b\left(F''v + G''w^2\right) - 2g^{ab}\left(F'v + G'w^2\right)$$
$$+4\left(\beta^a B^b + B^a\beta^b\right)G'w + 2B^aB^bG$$

$$\frac{\partial \Xi^a}{\partial B_b} = 4\beta^a\beta^b G'w - 2g^{ab}Gw + 4\beta^a B^b F' + 2B^a\beta^b G \tag{57}$$

At equilibrium

$$\frac{\partial^2 \Xi^a}{\partial \beta_c \partial \beta_b} = \frac{\partial^2 \Xi^a}{\partial B_c \partial \beta_b} = 0 \tag{58}$$

and

$$\frac{\partial^2 \Xi^a}{\partial B_c \partial B_b} = -4\beta^a_{(eq)}\beta^b_{(eq)}\beta^c_{(eq)}G' + 2\left(g^{ab}\beta^c_{(eq)} + g^{ac}\beta^b_{(eq)}\right)G + 4\beta^a_{(eq)}g^{bc}F' \tag{59}$$

Causality demands that

$$\begin{pmatrix} \frac{\beta^a_{(eq)}v_a}{\beta^2_{(c)}} & 0 & 0 & 0 \\ 0 & \frac{\partial \chi^a_c}{\partial \beta_c \partial \beta_b}v_a & 0 & 0 \\ 0 & 0 & \frac{\partial\left(\chi^a_q + \Xi^a\right)}{\partial B_c \partial B_b}v_a & 0 \\ 0 & 0 & 0 & \beta^a_{(eq)}v_a \end{pmatrix} \tag{60}$$

should be negative-definite for any future-oriented, timelike vector v^a . Since χ^a_q represents a perfect fluid, we obtain causality under the usual conditions [1]. We only need to verify that $\frac{\partial \Xi^a}{\partial B_c \partial B_b}v_a$ is negative-definite by itself. Specifically, we want to know if

$$\frac{\partial \Xi^a}{\partial B_c \partial B_b}v_a = 4\beta^a_{(eq)}v_a\left(-\beta^b_{(eq)}\beta^c_{(eq)}G' + g^{bc}F\prime\right) + 2\left(v^b\beta^c_{(eq)} + v^c\beta^b_{(eq)}\right)G \tag{61}$$

is negative-definite, that is if $L_b L_c \frac{\partial(\Xi^a)}{\partial B_c \partial B_b}v_a < 0$ for any L_a. To achieve this, let's decompose L_b into its longitudinal and transverse part relative to $\beta_{(eq)\,b}$:

$$L_b = l\left(\frac{\beta_{(eq)\,b}}{u}\right) + R_b \qquad ; \qquad R^b\beta_{(eq)\,b} = 0 \tag{62}$$

We easily work out the following

$$4\beta^a_{(eq)}v_a\left(-G' - \frac{(F'+G)}{u}\right) l^2 + 4\beta^a_{(eq)}v_a R_b R^b F' - 4R_b v^b G\, l \tag{63}$$

We can already extract some information. Taking $l=0$ we have

$$\beta^a_{(eq)}v_a R_b R^b F' < 0 \tag{64}$$

which imply that $F'>0$ since $\beta^a_{(eq)}v_a < 0$ and $R_b R^b > 0$. Also, Taking $R_b = 0$ we obtain

$$\beta^a_{(eq)}v_a\left(-G' - \frac{(F'+G)}{u}\right) l^2 < 0 \tag{65}$$

this time implying that $\left(-G' - \frac{(F'+G)}{u}\right) > 0$. Let us now write

$$v^b = \lambda\beta^b_{(eq)} + \omega^b \qquad ; \qquad \beta_{(eq)\,b}\omega^b = 0 \tag{66}$$

and decompose the spatial vector R_b into the part which is longitudinal and transversal to ω^b:

$$R^b = \eta\omega^b + t^b \qquad ; \qquad t_b\beta^b_{(eq)} = 0 = \omega_b t^b \tag{67}$$

Therefore $R_b v^b = \eta\omega_b\omega^b$ and, replacing in (61), we obtain upon division by $\beta^a_{(eq)}v_a < 0$:

$$\left(-G' - \frac{(F'+G)}{u}\right) l^2 - \frac{\omega_b\omega^b}{\xi^a v_a}G\eta l + \left(\eta^2\omega_b\omega^b + t_b t^b\right) F' \tag{68}$$

which should be now positive-definite in order to have (61) negative-definite. Since $t_b t^b F' > 0$ it suffice then to ask

$$\left(-G' - \frac{(F'+G)}{u}\right) l^2 - \frac{\omega_b\omega^b}{\beta^a_{(eq)}v_a}G\eta l + \eta^2\omega_b\omega^b F' > 0 \tag{69}$$

This is a quadratic form in l and η and we are thus demanding that the matrix

$$\begin{pmatrix} \left(-G' - \frac{(F'+G)}{u}\right) & -\frac{\omega_b\omega^b}{2\beta^a_{(eq)}v_a}G \\ -\frac{\omega_b\omega^b}{2\beta^a_{(eq)}v_a}G & \omega_b\omega^b F' \end{pmatrix} \tag{70}$$

should be positive-definite. We already know that this means that the diagonal element are positive, a fact that we already deduced. Now since the matrix is real and symmetric, we know that its eigenvalues are real. Since the diagonal elements are positive, then it is sufficient to prove that the determinant is positive. The determinant is given by

$$\omega_b\omega^b\left[\left(-G' - \frac{(F'+G)}{u}\right) F' - \frac{\omega_b\omega^b}{4\left(\beta^a_{(eq)}v_a\right)^2}G^2\right] > 0 \tag{71}$$

This condition can be simplified by considering that $v_a v^a = \lambda^2 \beta_{(eq)\,a}\beta^a_{(eq)} + \omega_a\omega^a < 0$ that is, using $\beta_{(eq)\,a}\beta^a_{(eq)} = -u$

$$\omega_a\omega^a < \lambda^2 u \tag{72}$$

Also $\beta^a_{(eq)} v_a = -\lambda u$. Therefore

$$\frac{\omega_a\omega^a}{\left(\beta^a_{(eq)} v_a\right)^2} = \frac{\omega_a\omega^a}{\lambda^2\mu^4} < \frac{1}{u} \tag{73}$$

Then if

$$\left(-G' - \frac{(F'+G)}{u}\right) F' > \frac{G^2}{4}\frac{1}{u} \tag{74}$$

then condition (71) is fulfilled. That is, the matrix (**??**) is positive definite if,

$$F' > 0 \tag{75}$$

$$\left(G' + \frac{(F'+G)}{u}\right) < 0 \tag{76}$$

and

$$\left(G' + \frac{(F'+G)}{u}\right) F' < -\frac{G^2}{4u} \tag{77}$$

To see the meaning of these conditions, consider the case where F and G follow power laws. Dimensional analysis in natural units ($c = \hbar = k_B = 1$) indicates that

$$[\Xi] = T^2 \tag{78}$$

Thus, writing $F = F_0 u^{\alpha+1}$ and $G = G_0 u^{\alpha}$ implies that $\alpha = -3$, that is

$$\Xi = F_0 u^{-2} v + G_0 u^{-3} w^2 \tag{79}$$

The three conditions (**??**),(**??**) and (**??**) read

$$F_0 < 0$$

$$G_0 + F_0 > 0 \ \Rightarrow G_0 > -F_0 > 0$$

$$4F_0\left(G_0 + F_0\right) < -\frac{1}{4}G_0^2 \tag{80}$$

Substituting $F_0/G_0 \equiv -x$, we have the following inequalities for x :

$$4x^2 - 4x + \frac{1}{4} < 0 \tag{81}$$

implying that $1/2 - \sqrt{3}/4 < x < 1/2 + \sqrt{3}/4$. Observe that both limit cases $x = 0$ and $x = 1$ are excluded.

We thus obtain the following final form for Ξ

$$\Xi = \frac{\Omega_0}{u}\left[x\left(\frac{v}{u}\right) - \left(\frac{w}{u}\right)^2\right] \tag{82}$$

($\Omega_0 = -G_0 < 0$) which can be rewritten in a covariant manner as (writing $\mu \equiv \sqrt{-\beta_a\beta^a}$)

$$\Xi = \frac{\Omega_0}{\mu^4}B^aB^b\left[-x\,g_{ab} - \frac{\beta_a\beta_b}{\mu^2}\right] \tag{83}$$

A. Nonlinear approach to equilibrium in the homogeneous case

We shall conclude this paper by studying the nonlinear evolution of the simple model just described. By simplicity, we shall work in flat space time and assume a homogeneous model, implying in particular that β^a and B^a are colinear, $\beta^a = \beta u^a$ and $B^a = Bu^a$. We choose a frame at rest with the fluid where $u^a = \delta^a_0$.

The equations of motion are

$$j^a_{;a} = V'(\phi) + \Delta \tag{84}$$

$$T^{ab}_{+;a} = 0 \tag{85}$$

$$T^{ab}_{-;a} = 2I^b \tag{86}$$

$$\beta^a_{(c)}\phi_{,a} = -\xi \tag{87}$$

Lets introduce the two energy-momentum tensor of both (perfect) fluids without interaction

$$T^{ab}_{c0} = \frac{\xi^2}{\mu^4}\beta^a_c\beta^b_c + g^{ab}\left(\frac{1}{2}\frac{\xi^2}{\mu^2_c} - V(\phi)\right)$$

and

$$T_{q0}^{ab} = g^{ab} p_q + u^a u^b \left(\rho_q + p_q\right)$$

The interaction between the two is given by ($\beta = \frac{1}{2}(\beta_c + \beta_q)$ and $B = \beta_c - \beta_q$)

$$T_{\Xi +}^{ab} = \frac{\partial \Xi^a}{\partial \beta_b}$$

and

$$T_{\Xi -}^{ab} = 2\frac{\partial \Xi^a}{\partial B_b}$$

We use as model for our interaction term

$$\Xi = \frac{B^2}{\beta^4}\left(F_0 + G_0\right) \equiv \frac{B^2}{\beta^4}\Gamma$$

Under the simplifying hypothesis stated above we have

$$T_{c0}^{00} = \frac{1}{2}\frac{\xi^2}{\beta_c^2} + V\left(\phi\right)$$

and

$$\frac{\partial \Xi^0}{\partial \beta_0} = 20\frac{B^2}{\beta^6}\Gamma$$

$$\frac{\partial \Xi^0}{\partial B_0} = -8\frac{B}{\beta^5}\Gamma$$

It is convenient to introduce a new variable s defined as

$$B = \beta_c - \beta_q = s\beta_q$$

implying

$$\beta = \frac{\beta_c + \beta_q}{2} = \left(\frac{s}{2} + 1\right)\beta_q$$

and to express the equations in terms of the canonical momentum

$$\pi = \phi_{,t} = -\frac{\xi}{\beta_c}$$

The energy-momentum tensors thus read

$$T_{+}^{00} = \frac{\pi^2}{2} + V(\phi) + \left[\sigma + 20\Gamma\frac{s^2}{(1+s/2)^6}\right]\frac{1}{\beta_q^4}$$

$$T_{-}^{00} = \frac{\pi^2}{2} + V(\phi) - \left[\sigma + 16\Gamma \frac{s}{(1+s/2)^5}\right] \frac{1}{\beta_q^4}$$

We are left with four O.D.E.

$$-\pi_{,t} = V'(\phi) + \Delta$$

$$\left(T_{+}^{00}\right)_{,t} = 0$$

$$\left(T_{-}^{00}\right)_{,t} = 2I^0$$

$$\phi_{,t} = \pi$$

where

$$\Delta = M^{\xi\xi}\xi - 2B(1-\kappa) M^{\xi 0}$$

and

$$I^0 = -BM^{00} + 2\kappa\xi M^{\xi 0}$$

The dimensions of the M^{AB} terms are

$$\begin{aligned} \left[M^{\xi\xi}\right] &= T^2 \\ \left[M^{00}\right] &= T^6 \\ \left[M^{\xi 0}\right] &= T^4 \\ [\xi] &= T \end{aligned}$$

This validates the following modelling for the M^{AB}

$$\begin{aligned} M^{\xi\xi} &= -\frac{K_0}{\tau^2} \\ M^{00} &= -\frac{L_0}{\tau^6} \\ M^{\xi 0} &= \frac{M_0}{\tau^4} \end{aligned}$$

where τ is some function of the dynamical variables with dimensions T^{-1} and $K_0, L_0 > 0$. Moreover, positive entropy production imposes the following relationship:

$$K_0 L_0 \geq M_0^2$$

We will also take the q -fluid as describing radiation so that $\rho = \sigma T^4$ with the Stefan - Boltzmann constant $\sigma > 0$. The equation (84) becomes

$$-\pi_{,t} = V'(\phi) + \frac{K_0}{\tau^2}(s+1)\beta_q \pi - 2(1-\kappa)s\beta_q \frac{M_0}{\tau^4}$$

Observe that

$$\begin{aligned}\left[\frac{1}{2}\pi^2 + V(\phi)\right]_{,t} &= \pi\left[\pi_{,t} + V'(\phi)\right] \\ &= \pi\left[-\frac{K_0}{\tau^2}(s+1)\beta_q\pi + 2B(1-\kappa)\frac{M_0}{\tau^4}\right]\end{aligned}$$

$$I^0 = L_0 \frac{s}{\tau^6}\beta_q - \kappa(s+1)\frac{\beta_q}{\tau^4}\pi M_0$$

And define

$$K = K_0(s+1)\frac{\beta_q}{\tau^2}\pi - 2(1-\kappa)s\beta_q\frac{M_0}{\tau^4}$$

$$c_1 = \sigma + 16\Gamma\frac{s}{(1+s/2)^5}$$

$$c_2 = \sigma + 20\Gamma\frac{s^2}{(1+s/2)^6}$$

Then

$$\pi_{,t} = -V'(\phi) - K$$

$$-K\pi - c_2\frac{4}{\beta_q^5}\beta_{q,t} + \frac{40\Gamma}{\beta_q^4}\frac{s(1-s)}{(1+s/2)^7}s_{,t} = 0$$

and

$$-K\pi + c_1\frac{4}{\beta_q^5}\beta_{q,t} - \frac{16\Gamma}{\beta_q^4}\frac{(1-2s)}{(1+s/2)^6}s_{,t} = 2I^0$$

The final system of equations is

$$s_{,t} = \frac{\beta_q^4(1+s/2)^6}{8\Gamma c_3}\left\{2I^0 c_2 + K\pi(c_1+c_2)\right\}$$

$$\beta_{q,t} = \frac{\beta_q^5}{4c_3}\left\{10\frac{s(1-s)}{1+s/2}I^0 + K\pi h(s)\right\}$$

$$\pi_{,t} = -V'(\phi) - K$$

$$\phi_{,t} = \pi$$

where

$$c_3 = 5\frac{s\,(1-s)}{(1+s/2)}c_1 - 2\,(1-2s)\,c_2$$

$$h\,(s) = 2\,(1-2s) + \frac{5(1-s)s}{1+s/2}$$

$$V\,(\phi) = \frac{1}{2}m^2\phi^2 + \lambda\phi^4$$

Finally

$$T_+^{00} = \frac{1}{2}\pi^2 + V\,(\phi) + \frac{c_2}{\beta_q^4}$$

is both positive definite and conserved.

These equations generally describe the approach to equilibrium. This is most clearly seen in the limit where s remains small. In this limit, we have $c_1 \sim c_2 \sim \sigma$, $c_3 \sim -2\sigma < 0$, $h\,(s) \sim 2$ and $K \sim K_0\beta_q\pi/\tau^2$. In particular, the equation for the field ϕ describes damped oscillations, but the damping "constant" is truly a dynamical variable, thus opening a mechanism to include memory effects in the dynamics and interesting behavior; note by exemple that K can change sign if the coupling term $M_0 \neq 0$.

More generally, by exploring parameter space we find a variety of behaviors. In particular, the approach to equilibrium may be either over or underdamped; this second case seems to be relevant to the description of preheating episodes.

V. FINAL REMARKS

The main conclusion of this work is that the conditions of causality, stability and a proper thermodynamic behavior put concrete limits on possible phenomenological models of the reheating period. We have shown concrete examples of divergence type theories which satisfy these requirements. Unlike earlier work, we have described the inflaton as a field, rather than disregarding its coherence by describing it simply as another fluid. This has required an extension both of the usual Klein - Gordon and DTT frameworks. We have also shown how this field can be consistently coupled to a fluid.

The equations of motion we have derived in this last Section show also that it is possible to explore the bulk of possible dynamical behaviors already with models with a minimal set of undetermined parameters (in our case, these were Γ, K_0, L_0, M_0 and the functional form of τ). These parameters may be estimated by fitting the predictions of the model to microscopic calculations in controlled limiting cases, much in the same way as viscosity coefficients in field theory are computed by analyzing the damping of extra long wavelength fluctuations [20–25]. We may then obtain reliable phenomenological models to use as a tool to explore the full nonlinear physics of reheating, with an enormous gain in simplicity as compared to a full attack from a first principles perspective.

VI. APPENDIX: DIVERGENCE TYPE THEORIES

Following Geroch [1], divergence type theories are usually described in terms of some tensorial quantities that obey conservation equations

$$T^{ab}_{;b} = 0$$

$$N^{a}_{;a} = 0$$

$$A^{abc}{}_{;a} = I^{bc}$$

This is a simple and slight generalization of relativistic fluid theories proposed initially by Liu, Muller and Ruggeri [52]. In this setting, T^{ab} is the energy-momentum tensor and N^a is the particle current. Their corresponding equation simply expresses conservation of energy, momentum and mass. The third equation will describe the dissipative part. The energy-momentum tensor is symmetric and $A^{abc} = A^{acb}$; $A^{ab}{}_{b} = 0$ and $I^{a}_{;a} = 0$. The entropy current is enlarged to read

$$S^a = \chi^a - \xi_b T^{ab} - \xi N^a - \xi_{bc} A^{abc}$$

The ξ, ξ^a, ξ^{ab} are the dynamical degrees of freedom. The following relations hold [1]

$$N^a = \frac{\partial \chi^a}{\partial \xi}$$

$$T^{ab} = \frac{\partial \chi^a}{\partial \xi_b}$$

$$A^{abc} = \frac{\partial \chi^a}{\partial \xi_{bc}}$$

Symmetry of the energy-momentum tensor implies that

$$\chi^a = \frac{\partial \chi}{\partial \xi_a}$$

That is all the fundamental tensors of the theory can be obtained from the generating functional χ. The entropy production is given by

$$S^a_{\ ;a} = -I^{bc}\xi_{bc}$$

Positive entropy production is ensured by demanding that $I^{bc} = M^{(bc)(de)}\xi_{de}$, where M is negative definite.

Ideal fluids are an important if somewhat trivial example. To obtain ideal hydrodynamics within the DTT framework, consider a generating functional $\chi_p = \chi_p(\xi, \mu)$ where $\mu \equiv \sqrt{-\xi_a \xi^a}$. It is a simple matter to obtain

$$\chi_p^a = -\frac{\xi^a}{\mu}\frac{\partial \chi_p}{\partial \mu}$$

$$T_p^{ab} = -\frac{g^{ab}}{\mu}\frac{\partial \chi_p}{\partial \mu} + \frac{\xi^a \xi^b}{\mu^2}\left[-\frac{1}{\mu}\frac{\partial \chi_p}{\partial \mu} + \frac{\partial^2 \chi_p}{\partial \mu^2}\right]$$

A simple comparison with the perfect fluid form of the energy-momentum tensor $T^{ab} = g^{ab}p + u^a u^b [p + \rho]$ implies the following identification

$$p = -\frac{1}{\mu}\frac{\partial \chi_p}{\partial \mu} \tag{88}$$

$$\rho = \frac{\partial^2 \chi_p}{\partial \mu^2}$$

Note that the conserved current can be quite generally written as

$$N^a = \frac{\partial}{\partial \xi}\left(-\frac{\xi^a}{\mu}\frac{\partial \chi}{\partial \mu}\right) = \xi^a \frac{\partial p}{\partial \xi} \tag{89}$$

A less trivial but important example both historically and conceptually is the Eckart theory which can be obtained from [1]

$$\chi_E = \chi_p + \frac{1}{2}\zeta_{ab}u^a u^b$$

Performing a Legendre transform to the new variables $\xi\,\xi^a, \xi^{ab}$ one obtains a system of first order differential equations of the form

$$\frac{\partial^2 \chi^a}{\partial \xi_A \partial \xi_B} \xi_{B;a} = I^A$$

where ξ_A stand for the entire collection of variables (ξ, ξ_a, ξ_{ab}) and similarly $I^A \equiv \left(0, 0, I^{ab}\right)$ represent the dissipative source; the index A thus covers 14 dimensions in our example. This first order system of differential equations is symmetric since

$$\frac{\partial^2 \chi^a}{\partial \xi_A \partial \xi_B} = \frac{\partial^2 \chi^a}{\partial \xi_B \partial \xi_A}$$

Note that we have a system of the form

$$A^i v_{,i} + Bv = 0$$

where i is a space-time index, the A^i and B are $k \times k$ matrices and v is a k -vector. Now this (first order) system is hyperbolic if all its eigenvalues are real; each of these eigenvalues represent the velocity of propagation of some small disturbance in space. These in turn propagate along hypersurfaces called characteristics whose existence is insured by the existence of k real eigenvalues [53,54]. If the matrices A^i and B are symmetric then it suffices that some combination $A^i v_i$ be definite (negative-definite given our choice of the signature for the metric) to insure that all the eigenvalues are real (but some could be degenerate). An usual case happens when this combination reduces to A^0 , the vector v being the time-like vector $\left(1, \overrightarrow{0}\right)$. In a relativistic theory one would expect hyperbolicity to be invariant under (proper) Lorentz transformations; in this case we say the system is causal. In our context, one would thus say that the system is hyperbolic if

$$\frac{\partial^2 \chi^a}{\partial \xi_A \partial \xi_B} v_a$$

is negative-definite for some temporal vector v_a and the theory will be causal if this stay true for *any* temporal vector v_a.

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